

# Analytical formula for the roots of the general complex cubic polynomial

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## Abstract

We present a new method to calculate analytically the roots of the general complex polynomial of degree three. This method is based on the approach of clever change of variable involving an arbitrary parameters. The advantage of this method is that it gives the roots of the cubic polynomial as uniform formulae without multi-cases expressions. Also, it gives a criterion to determine the multiple roots. In contrast, the reference method for this problem (Cardan-Tartaglia) gives the roots of the cubic polynomial as multi-cases formulae according to the sign of its discriminant.

**Keywords:** cubic polynomial roots, clever change of variable, analytical uniform formulae, multiple root criterion.

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## 1. Introduction

The calculation of polynomial roots is very important branch in linear algebra. It is useful, for example, for the eigenvalues perturbation [1] and the solutions of system of nonlinear equations [2]. Moreover, it has many physical applications, as the study of the singularities of the surfaces of refractive indices in crystal optics [3] and the diagonal eigenvalues perturbation [4]. Another example is the calculation of the sound velocity anisotropy in cubic crystals [5], or the diagonalization of Christoffel tensor to calculate the velocities of the three quasi-modes of the elastic waves in anisotropic medium [6]. An explicit calculation for particular case of  $3 \times 3$  real symmetric matrix has studied, see Ref. [7].

Cardan and Tartaglia gave a method to calculate the roots of the general polynomial of degree three as a linear combination of two cubic roots [8]. But, the inconvenience of their method is that the coefficients of this linear combination have a multi-cases formulae according to the sign of the discriminant of the cubic polynomial [9, 10]. Consequently, in the numerical implementations, we need to make a multiple conditions for the different executing statements. Effectively, a limited number of multiple executing statements does not require an important time. Since the calculation of the roots of cubic polynomial is repetitive and often required, then it will be important to use an uniform formulae in order to avoid the multiple executing statements for each calculation.

In this paper, we present an uniform analytical formulae for the roots of the general complex cubic polynomial, with a different method than that of Cardan-Tartaglia. Consequently, we can accelerate the numerical implementations by averting the multiple executing statements. The difference between our formulae and these of Cardan-Tartaglia is that we give an analytical expressions for the coefficients of the linear combination for the roots. Moreover, we present a criterion to determine the case where a cubic polynomial has a multiple root.

Our method is based on the approach of clever change of variable involving an arbitrary parameters, which rests on two principal ideas to diagonalize the general  $3 \times 3$  complex matrix : Firstly, we construct one or several clever change of variable involving an arbitrary parameters. So, the characteristic polynomial of this matrix becomes equivalent to another polynomial equation in terms of a new variable, where the coefficients of the new polynomial equation depend on the arbitrary parameters. Secondly, we choose these arbitrary parameters as required to make the form of the new polynomial equation as the sum of a cube of certain monomial and a term independent of the unknown new variable. Therefore, we can solve easily this new polynomial equation and consequently deduce the solution of the original equation, which is the eigenvalues of the general  $3 \times 3$  complex matrix.

In the sequel, we apply this approach to calculate the roots of the general complex cubic polynomial. In fact, we construct a particular matrix such that its characteristic polynomial will coincide with this cubic polynomial. Consequently, the eigenvalues of this particular matrix become identical to the roots of this cubic polynomial.

The rest of the paper is organized as follows. In section 2, we derive the approach of clever change of variable involving an arbitrary parameters. Consequently in section 3, we apply it on the general complex cubic polynomial.

## 2. Approach of clever change of variable involving an arbitrary parameters

Let  $\mathbf{M}$  be the general  $3 \times 3$  complex matrix :

$$\mathbf{M} := \begin{pmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{pmatrix}.$$

In this section, we aim to calculate the spectrum of  $\mathbf{M}$  with this approach. We detail it by following four steps. The first step consists in taking a change of variable for the eigenvalues of  $\mathbf{M}$  as follows :

**Proposition 2.1.** *An eigenvalue  $p$  of matrix  $\mathbf{M}$  can be written in terms of two parameters  $\tilde{a}$  and  $\tilde{b}$  as follows :*

$$p = \frac{1}{2} \left[ \tilde{a} - \tilde{b} + m_1 + m_5 + m_3 m_8 m_2^{-1} \mp \sqrt{(\tilde{a} - \tilde{b} - m_1 + m_5 + m_3 m_8 m_2^{-1})^2 + 4(m_2 m_4 + m_3 m_7)} \right], \quad (1)$$

where  $\tilde{a}$  can be chosen arbitrary and  $\tilde{b}$  is governed by the following equation :

$$A + B\tilde{b} + C\tilde{b}^2 + D\sqrt{\Delta_p} + E\tilde{b}\sqrt{\Delta_p} + \frac{1}{2}(\tilde{b}^2\sqrt{\Delta_p} - \tilde{b}^3) = 0. \quad (2)$$

Here the terms of Eq. (2) depend on parameter  $\tilde{a}$  as follows :

$$\begin{aligned} A &= \frac{\tilde{a}^3}{2} + e_1 \tilde{a}^2 + e_2 \tilde{a} + e_3, & B &= -\frac{3\tilde{a}^2}{2} - 2e_1 \tilde{a} - e_2, & C &= \frac{3\tilde{a}}{2} + e_1, \\ D &= \frac{\tilde{a}^2}{2} + \left(e_1 + \frac{c_1}{2}\right) \tilde{a} + e_4, & E &= -\tilde{a} - \left(e_1 + \frac{c_1}{2}\right), & \Delta_p &= \tilde{b}^2 + 2(c_1 - \tilde{a})\tilde{b} + \tilde{a}^2 - 2c_1 \tilde{a} + c_2, \end{aligned} \quad (3)$$

where, the coefficients of Eq. (3) are defined in terms of the components of  $\mathbf{M}$  as follows :

$$\begin{aligned} c_1 &:= m_1 - m_5 - \frac{m_3 m_8}{m_2}, & c_2 &:= c_1^2 + 4(m_2 m_4 + m_3 m_7), \\ e_1 &:= \frac{3}{2}(m_1 - c_1) - \frac{1}{2}\text{Tr}(\mathbf{M}), & e_2 &:= \frac{3}{2}(m_1 - c_1)^2 + \frac{3}{8}(c_2 - c_1)^2 - (m_1 - c_1)\text{Tr}(\mathbf{M}) + \frac{1}{4}\{[\text{Tr}(\mathbf{M})]^2 - \text{Tr}(\mathbf{M}^2)\}, \\ e_3 &:= \frac{1}{8}[(2m_1 - c_1)^3 + 3(2m_1 - c_1)c_2] - \frac{1}{4}[(2m_1 - c_1)^2 + c_2]\text{Tr}(\mathbf{M}) + \frac{1}{4}(2m_1 - c_1)\{[\text{Tr}(\mathbf{M})]^2 - \text{Tr}(\mathbf{M}^2)\} - \det(\mathbf{M}), \\ e_4 &:= \frac{1}{8}[3(2m_1 - c_1)^2 + c_2] - \frac{1}{2}(2m_1 - c_1)\text{Tr}(\mathbf{M}) + \frac{1}{4}\{[\text{Tr}(\mathbf{M})]^2 - \text{Tr}(\mathbf{M}^2)\}. \end{aligned} \quad (4)$$

$\text{Tr}(\mathbf{M})$  and  $\det(\mathbf{M})$  stand respectively for the trace and the determinant of matrix  $\mathbf{M}$ .

**Proof :**

Firstly, we decompose the matrix  $\mathbf{M}$  as the sum of two matrix  $\mathbf{M}_1$  and  $\mathbf{M}_2$  as follows :

$$\mathbf{M} = \underbrace{\begin{pmatrix} \tilde{a} & 0 & 0 \\ 0 & \tilde{b} & -m_0 \\ 0 & 0 & 0 \end{pmatrix}}_{\mathbf{M}_1} + \underbrace{\begin{pmatrix} m_1 - \tilde{a} & m_4 & m_7 \\ m_2 & m_5 - \tilde{b} & m_8 + m_0 \\ m_3 & m_6 & m_9 \end{pmatrix}}_{\mathbf{M}_2}, \quad (5)$$

where  $m_0$  is chosen such that  $\mathbf{M}_2$  has an eigenvector with this form  $\mathbf{v}_0 := [0, y_0, z_0]$ . Indeed, we find that if we take  $y_0/z_0 = -m_3/m_2$  and  $m_0 = (m_5 - \tilde{b} - k_0)m_3/m_2 - m_6$ , then the eigenvalue  $k_0$  of  $\mathbf{M}_2$  relative to  $\mathbf{v}_0$  will be given by :

$$k_0 = -m_3 m_8 m_2^{-1} + m_9. \quad (6)$$

Since we have from Eq. (6) one eigenvalue of  $\mathbf{M}_2$ , then we deduce that the two other eigenvalues of  $\mathbf{M}_2$  are given by :

$$k = \frac{1}{2} \left[ -\tilde{a} - \tilde{b} + m_1 + m_5 + m_3 m_8 m_2^{-1} \mp \sqrt{(\tilde{a} - \tilde{b} - m_1 + m_5 + m_3 m_8 m_2^{-1})^2 + 4(m_2 m_4 + m_3 m_7)} \right]. \quad (7)$$

Secondly, we search an eigenvector for  $\mathbf{M}$  with the form  $\mathbf{v} + \mathbf{v}_0$ , where  $\mathbf{v} := [x, y, z]$  is an eigenvector of  $\mathbf{M}_2$  relative to  $k$ . So, we obtain from Eq. (5) that the vectorial equation

$$\mathbf{M}(\mathbf{v}_0 + \mathbf{v}) = p(\mathbf{v}_0 + \mathbf{v}); \quad p \in \mathbb{C}.$$

is equivalent to the following system :

$$(S) \begin{cases} x(\tilde{a} + k) = xp, \\ \tilde{b}(y_0 + y) - m_0(z_0 + z) + k_0 y_0 + ky = p(y_0 + y), \\ k_0 z_0 + kz = p(z_0 + z). \end{cases}$$

We remark that if we take  $p = \tilde{a} + k$ , then the second and third equations of (S) are verified from the fact that  $\mathbf{v}_0$  and  $\mathbf{v}$  are respectively two eigenvectors of  $\mathbf{M}_2$  relative to its eigenvalues  $k_0$  and  $k$ . So, (S) is equivalent to the single equation  $p = \tilde{a} + k$ . Consequently, Eq. (7) implies that  $p$  depends only on  $\tilde{a} - \tilde{b}$  like in Eq. (1). Therefore, by inserting Eq. (1) in the characteristic equation of  $\mathbf{M}$  which is given by

$$p^3 - \text{Tr}(\mathbf{M})p^2 + \frac{1}{2}[\text{Tr}(\mathbf{M})^2 - \text{Tr}(\mathbf{M}^2)]p - \det(\mathbf{M}) = 0, \quad (8)$$

we can choose arbitrary among  $\tilde{a}$  and  $\tilde{b}$  one parameter as required, while the other parameter will be the unique unknown of Eq. (8), as well as  $p$  is an eigenvalue of  $\mathbf{M}$ . To end the proof, we develop Eq. (8) to deduce Eq. (2).  $\square$

Notice that the choice of the sign before the square root in Eq. (1) leads finally to the same result. So, in the sequel we consider the positive sign.

In the second step, we aim to simplify Eq. (2) satisfied by the unknown  $\tilde{b}$ , where  $\tilde{a}$  can be chosen as required. So, we derive from Eq. (2) two equations of  $\tilde{b}$  given by the following two propositions :

**Proposition 2.2.** *For fixed  $\tilde{a}$ ,  $\tilde{b}$  satisfies the following equation :*

$$d_1 + d_2 \tilde{b} + d_3 \tilde{b}^2 + d_4 \sqrt{\Delta_p} - d_3 \tilde{b} \sqrt{\Delta_p} = 0, \quad (9)$$

where the coefficients of Eq. (9) are given by :

$$d_1 := d_3 \tilde{a}^2 + n_1 \tilde{a} + n_9, \quad d_2 := -2d_3 \tilde{a} - n_1, \quad d_3 := m_2 m_4 + m_3 m_7, \quad d_4 := d_3 \tilde{a} + n_{10}.$$

Here  $n_1$ ,  $n_9$  and  $n_{10}$  are defined in terms of the coefficients of Eq. (4) as follows :

$$n_1 := \frac{c_1 c_2}{2} + c_1 e_2 + c_2 c_1 - 2c_1 e_4 - e_3, \quad n_9 := c_1 e_3 + c_2 e_4, \quad n_{10} := e_3 + c_1 e_4. \quad (10)$$

**Proof :**

On one hand we multiply Eq. (2) by  $\tilde{b}$  and on the other hand we multiply it by  $\sqrt{\Delta_p}$ . Consequently, we sum the obtained equations to deduce Eq. (9).  $\square$

**Proposition 2.3.** *For fixed  $\tilde{a}$ ,  $\tilde{b}$  satisfies the following equation :*

$$f_1 + f_2 \tilde{b} + f_3 \tilde{b}^2 + f_4 \sqrt{\Delta_p} + f_3 \tilde{b} \sqrt{\Delta_p} = 0, \quad (11)$$

where the coefficients of Eq. (11) are given by :

$$\begin{aligned} f_1 &:= f_3 \tilde{a}^2 + n_2 \tilde{a} + n_3, & f_2 &:= -2f_3 \tilde{a} - n_2, \\ f_3 &:= (m_5 - m_1) m_3 m_7 + (m_9 - m_1) m_2 m_4 - m_2 m_6 m_7 - m_3 m_4 m_8, & f_4 &:= -f_3 \tilde{a} + n_8. \end{aligned}$$

Here  $n_2$  and  $n_3$  are defined in terms of the coefficients of Eq. (4) as follows :

$$n_2 := c_2 d_3 - 2c_1^2 e_4 - 2c_1 e_3 - 2d_3 e_2, \quad n_3 := c_2 e_3 - 2d_3 e_3 + c_1 e_2 e_4, \quad n_8 := c_1 e_3 + e_2 e_4 - 2d_3 e_4. \quad (12)$$

**Proof :**

Similar to the proof of proposition 2.2, on one hand we multiply Eq. (2) by  $2d_3$  and on the other hand we multiply Eq. (9) by  $\sqrt{\Delta_p}$ . Consequently, we subtract the obtained equations to deduce Eq. (11).  $\square$

Now, we show that equations (9) and (11) are together equivalent to equations (13) and (14) given by the following corollary :

**Corollary 2.4.** *For fixed  $\tilde{a}$ ,  $\tilde{b}$  satisfies the following equations :*

$$r_1 + r_2\tilde{b} + r_3\sqrt{\Delta_p} - 2\tilde{b}\sqrt{\Delta_p} = 0, \quad (13)$$

$$r_4 + r_5\tilde{b} + r_6\sqrt{\Delta_p} + 2\tilde{b}^2 = 0, \quad (14)$$

where the coefficients of Eq. (14) are given by :

$$\begin{aligned} r_1 &:= -r_2\tilde{a} + n_6, & r_2 &:= -\frac{n_1}{d_3} + \frac{n_2}{f_3}, & r_3 &:= 2\tilde{a} + n_7, \\ r_4 &:= 2\tilde{a}^2 + n_4\tilde{a} + n_5, & r_5 &:= -4\tilde{a} - n_4, & r_6 &:= \frac{n_{10}}{d_3} + \frac{n_8}{f_3} \end{aligned} \quad (15)$$

Here, the coefficients of Eq. (15) are defined in terms of the coefficients of Eq. (4) and Eq. (10) as follows :

$$n_4 := \frac{n_1}{d_3} + \frac{n_2}{d_3}, \quad n_5 := \frac{n_9}{d_3} + \frac{n_3}{f_3}, \quad n_6 := \frac{n_9}{d_3} - \frac{n_3}{f_3}, \quad n_7 := \frac{n_{10}}{d_3} - \frac{n_9 - 2d_3e_4}{f_3}.$$

**Proof :**

Equations (13) and (14) are given respectively by subtracting and summing Eq. (9) and Eq. (11).  $\square$

In the third step, we make the following change of variable for the unknown  $\tilde{b}$  :

$$l := -\left(\tilde{b} + o\sqrt{\Delta_p}\right) \quad (16)$$

in order to find a polynomial equation of  $\tilde{b}$ , where  $o$  is an arbitrary parameter can be chosen as required :

**Proposition 2.5.** *Let  $o \in \mathbb{C}$ . Then  $l$  verifies the following equation :*

$$2l^3 + C_l l^2 + B_l l + D_l = 0, \quad (17)$$

where the coefficients of Eq. (17) are given by :

$$\begin{aligned} B_l &:= l_3 o^2 - 2(l_1 \tilde{a} + l_4) o + 6\tilde{a}^2 - 2l_2 \tilde{a} + l_5, & C_l &:= -l_1 o + 6\tilde{a} - l_2, \\ D_l &:= -l_6 o^3 + (l_3 \tilde{a} - l_7) o^2 - (l_1 \tilde{a}^2 + 2l_4 \tilde{a} + l_8) o + 2\tilde{a}^3 - l_2 \tilde{a}^2 + l_5 \tilde{a} + l_9. \end{aligned}$$

Here  $B_l$ ,  $C_l$  and  $D_l$  are defined in terms of the coefficients of Eq. (4), Eq. (10) and Eq. (12) as follows :

$$\begin{aligned} l_1 &:= 2\frac{n_1}{d_3} - n_4 + r_6, & l_2 &:= r_6 - n_4 - 2\frac{n_{10}}{d_3}, & l_3 &:= n_5 - 2c_2 + 2c_1 r_6 + \frac{r_6 n_1 - n_4 n_{10}}{d_3}, \\ l_4 &:= 2\frac{n_9}{d_3} - n_5 - c_1 r_6, & l_5 &:= n_5 + \frac{n_4 n_{10} - r_6 n_1}{d_3}, & l_6 &:= 2c_1 n_5 + c_2 n_4 + \frac{n_1(n_5 - 2c_2) - (4c_1 + n_4)n_9}{d_3}, \\ l_7 &:= 2c_1 n_5 + c_2(n_4 + r_6) + \frac{(n_5 - 2c_2)n_{10} - r_6 n_9}{d_3}, & l_8 &:= c_2 r_6 + \frac{n_4 n_9 - n_1 n_5}{d_3}, & l_9 &:= \frac{n_5 n_{10} - r_6 n_9}{d_3}. \end{aligned}$$

**Proof :**

We have from Eq. (16) that

$$\tilde{b} = -\left(o\sqrt{\Delta_p} + l\right). \quad (18)$$

We take the square of Eq. (18) and we replace  $\Delta_p$  by its value given by Eq. (3). So, by inserting Eq. (18) in the obtained equation, we deduce that :

$$\tilde{b}^2 = (1 - o^2)^{-1} \left\{ [2ol - 2(c_1 - \tilde{a})o^3] \sqrt{\Delta_p} + (\tilde{a}^2 - 2c_1\tilde{a} + c_2)o^2 + l^2 - (c_1 - \tilde{a})o^2l \right\}. \quad (19)$$

Then, inserting Eq. (18) and Eq. (19) in Eq. (14), we obtain that :

$$\sqrt{\Delta_p} = \frac{(1 - o^2)(r_4 - r_5l) + 2[(\tilde{a}^2 - 2c_1\tilde{a} + c_2)o^2 - 2(c_1 - \tilde{a})o^2l + l^2]}{(1 - o^2)(r_5o - r_6) + 2[2(c_1 - \tilde{a})o^3 - 2ol]}. \quad (20)$$

Finally, Eq. (16) implies the following equation  $\tilde{b} + o\sqrt{\Delta_p} + l = 0$ . So, we take the square of it and afterwards insert Eq. (18) within. Therefore, to complete the proof, it is enough to insert Eq. (19) and Eq. (20) in the obtained equation and consequently to develop it.  $\square$

Since  $o$  is an arbitrary parameter, then we can choose  $o$  as required to solve equation (17) of the unknown  $l$  :

**Lemma 2.6.** *If  $o$  verifies the following polynomial equation of degree two :*

$$(l_1^2 - 6l_3)o^2 + 2(l_1l_2 + 6l_4)o + l_2^2 - 6l_5 = 0, \quad (21)$$

then  $l = -\tilde{a} + r_l$ , where

$$r_l = \frac{l_1o + l_2}{6} + ms_l,$$

for  $m \in \left\{ -1 ; \frac{1-\sqrt{-3}}{2} ; \frac{1+\sqrt{-3}}{2} \right\}$ . Here  $s_l$  is given by :

$$s_l := \sqrt[3]{\left(\frac{l_1o + l_2}{6}\right)^3} - \frac{1}{2}(l_6o^3 + l_7o^2 + l_8o - l_9).$$

**Proof :**

Equation (17) can be rewritten as follows

$$l^3 + 3\frac{C_l}{6}l^2 + 3\frac{B_l}{6}l + \left(\frac{C_l}{6}\right)^3 + s_l - \frac{(l_1^2 - 6l_3)o^2 + 2(l_1l_2 + 6l_4)o + l_2^2 - 6l_5}{12}\tilde{a} = 0. \quad (22)$$

The condition  $C_l^2 = 6B_l$ , which is equivalent to Eq. (21), allows us to factorise Eq. (22) as follows :

$$\left(l + \frac{C_l}{6}\right)^3 + s_l^3 = \left(l + \frac{C_l}{6} + s_l\right) \left[\left(l + \frac{C_l}{6}\right)^2 - s_l\left(l + \frac{C_l}{6}\right) + s_l^2\right] = 0. \quad (23)$$

So, Eq. (23) can be solved easily, wich ends the proofs.  $\square$

Lemma 2.6 gives the expressions for the set of solutions of  $l$ . Then, we deduce from the change of variable (18) the following polynomial equation of  $\tilde{b}$ , which will be useful to give an explicit formulae for the set of solutions of  $\tilde{b}$  :

**Corollary 2.7.** *If  $o$  verifies Eq. (21), then  $\tilde{b}$  verifies the following equation*

$$(1 - o^2)\tilde{b}^2 + 2[r_l - \tilde{a} + o^2(\tilde{a} - c_1)]\tilde{b} + (r_l - \tilde{a})^2 - o^2(\tilde{a}^2 - 2c_1\tilde{a} + c_2) = 0, \quad (24)$$

where

$$r_l \in \mathcal{S}_l := \left\{ \frac{l_1o + l_2}{6} - s_l ; \frac{l_1o + l_2}{6} + \frac{1 - \sqrt{-3}}{2}s_l ; \frac{l_1o + l_2}{6} + \frac{1 + \sqrt{-3}}{2}s_l \right\}.$$

**Proof :**

We have from Eq. (16) that

$$\sqrt{\Delta_p} = -\frac{1}{o}(\tilde{b} + l). \quad (25)$$

The proof ends by taking the square of Eq. (25) and afterwards replacing  $\Delta_p$  by its value given by Eq. (3) in the obtained equation.  $\square$

Finally, in the fourth step, we can determine the set of solutions of  $\tilde{b}$  by solving the nonlinear system constructed by Eq. (24) and one of the following equations (9), (11), (13) or (14). Then, we can solve this nonlinear system by choosing  $\tilde{a}$  as required to simplify the calculation. Consequently, we deduce from Eq. (1) the spectrum of  $\mathbf{M}$  :

**Corollary 2.8.** *The spectrum of matrix  $\mathbf{M}$  is given by :*

$$p = \frac{1}{2} \left[ \left( 1 + \frac{1}{o} \right) \frac{d_3 r_l^2 + n_{10}(1-o)r_l - n_9 o(1-o) - d_3 c_2 o^2}{(1-o)(n_{10} + n_1 o) - 2d_3 c_1 o^2 + d_3(1+o)r_l} + 2m_1 - c_1 - \frac{r_l}{o} \right], \quad (26)$$

where  $r_l \in \mathcal{S}_l$  and  $o$  verifies Eq. (21).

**Proof :**

Inserting Eq. (25) in both Eq. (9) and Eq. (24). Then, by substituting  $\tilde{b}^2$  from one obtained equation in the other, we deduce that :

$$\tilde{b} = \tilde{a} - \frac{d_3 r_l^2 + (1-o)n_{10}r_l - n_9 o(1-o) - d_3 c_2 o^2}{(1-o)(n_{10} + n_1 o) - 2d_3 c_1 o^2 + d_3(1+o)r_l}. \quad (27)$$

Finally, to deduce Eq. (26), we insert Eq. (25) in Eq. (1) and afterwards we insert Eq. (27) in the obtained equation.  $\square$

Moreover, in order to rationalize the denominator of Eq. (26) (see the proof of theorem 3.3), we calculate  $\tilde{b}$  and consequently deduce the spectrum of  $\mathbf{M}$  by another way than corollary 2.8 :

**Corollary 2.9.** *The spectrum of matrix  $\mathbf{M}$  is given by :*

$$p = \frac{1}{2} \left[ \left( 1 + \frac{1}{o} \right) \frac{n_6 o^2 - n_5 o + (r_6 - n_7 o)r_l}{(n_4 - n_7)o + r_2 o^2 + r_6 + 2or_l} + 2m_1 - c_1 - \frac{r_l}{o} \right], \quad (28)$$

where  $r_l \in \mathcal{S}_l$  and  $o$  verifies Eq. (21).

**Proof :**

We insert Eq. (25) in both Eq. (9) and Eq. (11). Then, by substituting  $\tilde{b}^2$  from one obtained equation in the other, we deduce that :

$$\tilde{b} = \tilde{a} - \frac{n_6 o^2 - n_5 o + (r_6 - n_7 o)r_l}{(n_4 - n_7)o + r_2 o^2 + r_6 + 2or_l}. \quad (29)$$

Finally, to deduce Eq. (28), we insert Eq. (25) in Eq. (1) and afterwards we insert Eq. (29) in the obtained equation.  $\square$

### 3. Application on the cubic polynomial

Let  $(\mathcal{P})$  be the general complex polynomial of degree three :

$$(\mathcal{P}) : x^3 + bx^2 + cx + d = 0; \quad b, c, d \in \mathbb{C}.$$

Now, we aim to calculate analytically the roots of  $(\mathcal{P})$  by applying the results of section 2. So, we introduce the following definitions which will be useful to find the expressions for the roots of  $(\mathcal{P})$  :

**Definition 3.1.** We associate to  $(\mathcal{P})$  the following quantities in terms of its coefficients :

$$\begin{aligned}\Delta_l &:= 2c^3(8b^6 + 132b^3d + 36d^2 + c^3 + 33b^2c^2 - 66bcd) + 12b^4c(d^2 - 7c^3) - b^2c^2d(24b^3 + 291d) + d^3(144bc - 2b^3 - 27d), \\ \Delta_o &:= -4b^3d + b^2c^2 + 18bcd - 4c^3 - 27d^2,\end{aligned}$$

and

$$d_o := 4b^4c^2 - 4b^3cd - 14b^2c^3 + b^2d^2 + 28bc^2d + c^4 - 12cd^2.$$

**Definition 3.2.** We associate to  $(\mathcal{P})$  the following quantities in terms of these given by definition 3.1 :

$$\delta_l := (d - bc)\sqrt{\Delta_o}(4b^2c^2 - 4bcd + 2c^3 + d^2) + \frac{\sqrt{-3}}{9}\Delta_l,$$

and

$$\begin{aligned}A_1 &:= -\frac{2\sqrt{-3}}{3}(4b^3c - 2db^2 - 13bc^2 + 15dc) + 2c\sqrt{\Delta_o}, \\ A_2 &:= 8b^5c^2 - 8b^4cd - 40b^3c^3 + 2b^3d^2 + 116b^2c^2d + 23bc^4 - 99bcd^2 - 21c^3d + 27d^3 - \sqrt{-3}(8b^2c^2 - 10bcd + c^3 + 3d^2)\sqrt{\Delta_o}.\end{aligned}$$

**Theorem 3.3.** The set of solutions for equation  $(\mathcal{P})$  is given in terms of two cubic roots  $R_1$  and  $R_1$  as follows :

$$x = \frac{\sqrt[3]{4}}{2}m \exp\left\{\sqrt{-1}\left[\arg(A_1\sqrt[3]{\delta_l}) - \arg(-d_oR_1)\right]\right\}R_1 + \frac{\sqrt[3]{4}}{2}m^2 \exp\left\{\sqrt{-1}\left[\arg(A_2\sqrt[3]{\delta_l^2}) - \arg(d_o^2R_2)\right]\right\}R_2 - \frac{b}{3}, \quad (30)$$

for  $m \in \left\{-1; \frac{1-\sqrt{-3}}{2}; \frac{1+\sqrt{-3}}{2}\right\}$ . Where,  $R_1$  and  $R_1$  are given by :

$$\begin{aligned}R_1 &:= \sqrt[3]{\frac{\sqrt{3}}{9}\sqrt{-\Delta_o} + \frac{2b^3 - 9cb + 27d}{27}}, \\ R_2 &:= \sqrt[3]{\frac{\sqrt{3}}{9}\sqrt{-\Delta_o} - \frac{2b^3 - 9cb + 27d}{27}}.\end{aligned}$$

Here,  $\Delta_o$ ,  $d_o$ ,  $\delta_l$ ,  $A_1$  and  $A_2$  are given by definitions 3.1 and 3.2.

**Proof :**

Firstly, we construct the following matrix

$$\mathbf{A} := \begin{pmatrix} -b & 1 & d^{-1}c \\ 0 & 0 & 1 \\ -d & 0 & 0 \end{pmatrix},$$

which has  $(\mathcal{P})$  as its characteristic polynomial. Consequently, the roots of  $(\mathcal{P})$  are identical to the spectrum of  $\mathbf{A}$ , which can be calculated from corollaries 2.8 and 2.9. But, these corollaries require firstly to make explicitly  $o$  and  $r_l$  relative to matrix  $\mathbf{A}$ . Then, by applying lemma 2.6 on matrix  $\mathbf{A}$ , we deduce that

$$o = d_o^{-1} \left[ b^2c^3 - c^4 - 3cd^2 + b^2d^2 - 2b^3cd \pm c(d - bc)\sqrt{-3\Delta_o} \right], \quad (31)$$

because Eq. (21) is a polynomial equation for  $o$  of degree two. In the sequel, we take the positive sign before  $c(d - bc)\sqrt{-3\Delta_o}$  in Eq. (31). Also, by applying lemma 2.6 on matrix  $\mathbf{A}$ , we deduce that

$$r_l = l_o - m\sqrt[3]{4}d_o^{-1}c\sqrt{\Delta_o}\sqrt[3]{\delta_l}.$$

Here,  $l_o$  is defined as follows :

$$l_o := d_o^{-1} \left[ 2b^4cd + b^3c^3 - b^3d^2 - 5b^2c^2d - 4bc^4 + 6bcd^2 + 5c^3d + \frac{\sqrt{-3}}{3}c(b^2c - 2bd - c^2)\sqrt{\Delta_o} \right].$$

Secondly, since we have the expressions of  $o$  and  $r_l$  relative to matrix  $\mathbf{A}$ , then by applying corollaries 2.8 and 2.9 on matrix  $\mathbf{A}$ , we deduce that the roots of  $(\mathcal{P})$  are given by :

$$\begin{aligned} x &= \frac{1}{2} \left[ \left( 1 + \frac{1}{o} \right) \frac{-cr_l^2 + d(1-o)r_l + bdo(1-o) + c(b^2 - 4c)o^2}{(1-o)[d + (bc - d)o] - 2bco^2 - c(1+o)r_l} - b - \frac{r_l}{o} \right] \\ &= \frac{1}{2} \left[ \left( 1 + \frac{1}{o} \right) \frac{bd(bc - d) + cd(b^2 - 2c) - d(2c^2 - bd)o^{-1} + do^{-1}(do^{-1} - d + 2bc)r_l}{2b^2c^2 - 4bcd - 2c^3 + d^2 - 2(c^3 - bcd + d^2)o^{-1} + d^2o^{-2} - 2c(d - bc)o^{-1}r_l} - b - \frac{r_l}{o} \right]. \end{aligned} \quad (32)$$

So, we aim to simplify formula (32). In fact, we deduce from (32) that

$$\begin{aligned} \frac{-cr_l^2 + d(1-o)r_l + bdo(1-o) + c(b^2 - 4c)o^2}{(1-o)[d + (bc - d)o] - 2bco^2 - c(1+o)r_l} &= \\ \frac{bd(bc - d) + cd(b^2 - 2c) - d(2c^2 - bd)o^{-1} + do^{-1}(do^{-1} - d + 2bc)r_l}{2b^2c^2 - 4bcd - 2c^3 + d^2 - 2(c^3 - bcd + d^2)o^{-1} + d^2o^{-2} - 2c(d - bc)o^{-1}r_l}. \end{aligned} \quad (33)$$

But, we have the following property, which allows us to remove the cube root derived from  $r_l$  in the denominator of the fractions of Eq. (33) :

$$\frac{\mathcal{A}}{\mathcal{B}} = \frac{C}{D} \Rightarrow \frac{\mathcal{A}}{\mathcal{B}} = \frac{C}{D} = \frac{\mathcal{M}\mathcal{A} + \mathcal{N}C}{\mathcal{M}\mathcal{B} + \mathcal{N}D}; \quad \forall \mathcal{A}, \mathcal{B}, C, D, \mathcal{M}, \mathcal{N} \in \mathbb{C}^*. \quad (34)$$

By applying property (34) on Eq. (33) for  $\mathcal{M} = 2o^{-1}c(d - bc)$  and  $\mathcal{N} = -c(1 + o)$ , we just obtain a square root in the consequent denominator. Therefore, it becomes easy to rationalize it and consequently to deduce that Eq. (32) is equivalent to the following equation :

$$x = -\frac{\sqrt[3]{4}}{8} \frac{md_oA_1\sqrt[3]{\delta_l} - m^2\sqrt[3]{4}A_2\sqrt[3]{\delta_l^2}}{d_o^2} - \frac{b}{3}. \quad (35)$$

Thirdly, in order to simplify  $d_o^2$  in the denominator of Eq. (35), we can prove that :

$$\begin{aligned} (A_1\sqrt[3]{\delta_l})^3 &= (-4d_oR_1)^3 \Leftrightarrow A_1\sqrt[3]{\delta_l} = \exp \left\{ \sqrt{-1} \left[ \arg(A_1\sqrt[3]{\delta_l}) - \arg(-4d_oR_1) \right] \right\} (-4d_oR_1), \\ (A_2\sqrt[3]{\delta_l^2})^3 &= (\sqrt[3]{4^2}d_o^2R_2)^3 \Leftrightarrow A_2\sqrt[3]{\delta_l^2} = \exp \left\{ \sqrt{-1} \left[ \arg(A_2\sqrt[3]{\delta_l^2}) - \arg(\sqrt[3]{4^2}d_o^2R_2) \right] \right\} \sqrt[3]{4^2}d_o^2R_2. \end{aligned} \quad (36)$$

Then, by inserting Eq. (36) in Eq. (35), we deduce Eq. (30). □

Finally, we introduce the following criterion, which allows us to determine where  $(\mathcal{P})$  has a double root :

**Criterion 3.4.**  $(\mathcal{P})$  has a double root if and only if

$$\left[ -d_oA_1 + (m_1 + m_2)\sqrt[3]{4}A_2\sqrt[3]{\delta_l} \right] \frac{\sqrt[3]{\delta_l}}{d_o^2} = 0,$$

for  $m_1 \neq m_2$  in  $\left\{ -1; \frac{1-\sqrt{-3}}{2}; \frac{1+\sqrt{-3}}{2} \right\}$ . Where, the double root is that relative to  $m_1$  and  $m_2$ .

**Proof :**

It is an immediate result from Eq. (35). □



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